

Braid group statistics implies scattering in three-dimensional local quantum physics

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Abstract

It is shown that particles with braid group statistics (Plektons) in three-dimensional space-time cannot be free, in a quite elementary sense: They must exhibit elastic two-particle scattering into every solid angle, and at every energy. This also implies that for such particles there cannot be any operators localized in wedge regions which create only single particle states from the vacuum and which are well-behaved under the space-time translations (so-called temperate polarization-free generators). These results considerably strengthen an earlier “NoGo-theorem for ‘free’ relativistic Anyons”.

As a by-product we extend a fact which is well-known in quantum field theory to the case of topological charges (i.e., charges localized in space-like cones) in $d \geq 4$, namely: If there is no elastic two-particle scattering into some arbitrarily small open solid angle element, then the 2-particle S-matrix is trivial.

1 Introduction

In relativistic quantum field theory in $d > 2$ space-time dimensions it is a well-established fact [1, 6] that if there is some arbitrarily small open solid angle element into which there is no elastic two-particle scattering, then the $2 \rightarrow 2$ particle S-matrix is trivial, and then there is also no particle production. We show that the same holds in the quite general setting of local quantum physics admitting topological charges in $d \geq 4$. Our main result, however, is that the hypothesis *cannot be satisfied* for Plektons in $d = 3$, namely: Braid group statistics implies elastic two-particle scattering. This also implies that for Plektons there cannot be a model which has temperate polarization-free generators in the sense of Borchers et al. [4]. (These are operators localized in wedge regions which create only single particle states from the vacuum and which are well-behaved under the space-time translations.) These results considerably strengthen an earlier “NoGo-theorem for ‘free’ relativistic Anyons” by the one of the authors [31].

In view of our finding that there are no free Plektons, the Lagrangean approach does not seem appropriate for the construction of relativistic quantum fields with

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braid group statistics, since its underlying strategy is the coupling of free fields to other fields or among themselves. In fact, the situation with respect to Lagrangean relativistic model building is quite unsatisfactory. Lagrangean local gauge theories have been proposed (for more than 20 years) where the matter fields assume anyonic properties by coupling to a Chern-Simons type gauge field [2, 24, 29, 40, 41]. However, except for lattice models [3, 27, 30], these models have not been made completely explicit. Consequently, as S. Forte put it [15, p. 235], different and contradictory conclusions have been drawn from the same starting point (Lagrangean).

We consider the scattering of two massive charged particles, assuming that the respective masses are isolated values in the mass spectra within the respective charge sectors. We admit that the particles carry topological charges, that is, charges which are localizable not in compact regions but only in space-like cones¹, which is the worst possible dis-locality in the purely massive case [9]. For simplicity, we assume that the masses of the two particles coincide. Given two single particle states ψ_1, ψ_2 in respective charge sectors π_1, π_2 , and having disjoint energy-momentum supports, Haag-Ruelle scattering theory associates an outgoing and an incoming 2-particle scattering state

$$\psi_1^{\text{out}} \times \psi_2, \quad \psi_1 \times \psi_2^{\text{in}}.$$

In the case of permutation group statistics, these scattering states depend only on the single particle states ψ_k , as indicated by the notation. However, in the case of braid group statistics they also depend on the space-time localization regions in which the charged states ψ_k have been “created from the vacuum”. Now suppose that there are sets of incoming momenta U_1, U_2 and of outgoing momenta V_1, V_2 on the mass shell, mutually disjoint:

$$U_1 \cap U_2 = V_1 \cap V_2 = U_i \cap V_j = \emptyset, \quad (1)$$

for which all scattering amplitudes in the channel $\pi_1 \times \pi_2 \rightarrow \pi_1 \times \pi_2$ are trivial — even though the corresponding process is admitted by energy-momentum conservation, namely

$$(U_1 + U_2) \cap (V_1 + V_2) \neq \emptyset. \quad (2)$$

More precisely, for all single-particle states ϕ_i in the sectors π_i with respective spectral supports in U_i and ψ_i in the same sectors π_i and with supports in V_i and for some admissible space-time localization regions² there holds

$$(\phi_1^{\text{in}} \times \phi_2, \psi_1^{\text{out}} \times \psi_2) = 0. \quad (3)$$

We then say that there is *no elastic two-particle scattering from $U_1 \times U_2$ into $V_1 \times V_2$ in the channel $\pi_1 \times \pi_2 \rightarrow \pi_1 \times \pi_2$* .³

Our first result is that, in space-time dimension $d \geq 4$, this hypothesis implies triviality of the two-particle S-matrix. For simplicity of the argument, we suppose that a pair of disjoint sets of incoming and outgoing momenta for which there is no scattering exists for every total momentum.

¹A space-like cone is a region in Minkowski space of the form $C = a + \cup_{\lambda > 0} \lambda \mathcal{O}$, where $a \in \mathbb{R}^4$ is the apex of C and \mathcal{O} is a double cone whose closure is causally separate from the origin.

²Only relevant in the case of braid group statistics.

³Contrapositively, we say that there *is* elastic two-particle scattering from $U_1 \times U_2$ into $V_1 \times V_2$ if for every admissible space-time localization region there are some ϕ_i, ψ_i such that the above scattering amplitude is non-zero.

Theorem 1 (Triviality of the 2-particle S-matrix in $d \geq 4$.) *Suppose that for every total momentum P , $P^2 \geq 4m^2$, there are non-empty open subsets U_1, U_2, V_1, V_2 of the mass shell, mutually disjoint, such that $P \in (U_1 + U_2) \cap (V_1 + V_2)$, and such that there is no elastic two-particle scattering from $U_1 \times U_2$ into $V_1 \times V_2$ in the channel $\pi_1 \times \pi_2 \rightarrow \pi_1 \times \pi_2$. Then the 2-particle S-matrix in this channel is trivial, i.e. for every pair of single particle states ψ_1, ψ_2 in the respective sectors π_1, π_2 the incoming and outgoing scattering states coincide:*

$$\psi_1 \times^{\text{in}} \psi_2 = \psi_1 \times^{\text{out}} \psi_2. \quad (4)$$

This extends a well-established fact from the case of compactly localized charges to the case of topological charges. We consider this result of interest in its own right in view of the various recent constructions of "non-local" quantum field theories [11, 20, 26]. For example, the deformation approach constructs theories with non-trivial S-matrices satisfying the hypothesis. Our theorem then gives a structural argument that in these models the spacelike-cone-algebras are too small as to satisfy the Reeh-Schlieder property — which otherwise has to be (and of course has been) checked by explicit calculations. The theorem is also needed in a recent algebraic version of the Jost-Schroer theorem [34].

Our main result concerns models with topological charges in three-dimensional space-time, where braid group statistics may occur. Here we consider the case where π_1 and π_2 are conjugate charges or, in the Abelian case, coincide. It turns out that the hypothesis (3) implies that the statistics parameter of the corresponding sector is real, thereby excluding braid group statistics. In other words, we show:

Theorem 2 (Braid group statistics implies scattering.) *Let π be a massive single particle representation with braid group statistics⁴, and let $\bar{\pi}$ be the conjugate sector. Then for any non-empty open sets of energy-momenta U_1, U_2, V_1, V_2 admitted by energy-momentum conservation (2) there is elastic two-particle scattering from $U_1 \times U_2$ into $V_1 \times V_2$ in the channel $\pi \times \bar{\pi} \rightarrow \pi \times \bar{\pi}$. In the case of Anyons, the same holds for the channel $\pi \times \pi \rightarrow \pi \times \pi$.*

Now Borchers et al. [4] show that the existence of temperate wedge-localized polarization-free generators (PFG's) implies that there are non-empty subsets U_i, V_i of the mass shell for which there is no elastic two-particle scattering even though admitted by energy-momentum conservation, see Eq. (3.27) in [4], which can be derived also in the braid group statistics case. We therefore conclude from the above theorem:

Corollary 3 (Non-existence of PFG's for Plektons.) *The existence of temperate wedge-localized polarization free generators excludes braid group statistics.*

We shall prove these theorems along the following standard lines. Due to the assumed disjointness of the energy-momentum supports, the state $\psi_1 \times^{\text{in}} \psi_2$ also has vanishing scalar product with $\phi_1 \times^{\text{in}} \phi_2$, and Eq. (3) can be rephrased by saying that the corresponding S-matrix element is trivial,

$$(\phi_1 \times^{\text{in}} \phi_2, \psi_1 \times^{\text{in}} \psi_2 - \psi_1 \times^{\text{out}} \psi_2) = 0. \quad (5)$$

⁴That is to say, with non-real statistics parameter, see Sec. 2

The first step is now to establish in the present setting the relevant LSZ relations, which relate the left hand side of the above equation with the amputated time-ordered function (35). The second step is to establish on-shell analyticity properties of the amputated time-ordered function. These analyticity properties are weaker [5] than in the case of compact localization but, as it turns out, still sufficient for the present purpose: To extend Eq. (5) to other particle configurations ψ_i (for fixed ϕ_i). Namely, in $d \geq 4$ Eq. (5) extends to *all* ψ_i , which leads to Theorem 1. In $d = 3$, it extends to such ψ_i which have common supports with the states ϕ_i , so that the states $\psi_1 \times^{\text{in}} \psi_2$ no longer have necessarily vanishing scalar products with $\phi_1 \times^{\text{in}} \phi_2$. Then a careful analysis of the dependence of the scattering states $\psi_1 \times^{\text{out}} \psi_2$, $\psi_1 \times^{\text{in}} \psi_2$ on the space-time localization regions shows that Eq. (5) is incompatible with braid group statistics.

The article is organized as follows. Section 2 is devoted to a more detailed account of the general setting, and of some known facts on the scattering of Plektons. In Section 3 we show that certain LSZ relations hold true in the case of topological charges, eventually with braid group statistics, under certain hypothesis on the momentum supports and space-time localization regions. In Section 4 we give a detailed account of the analyticity domain of the relevant amputated time-ordered (or advanced, or retarded) functions, see also [5]. Finally, we prove the two theorems in Section 5.

2 General Setting

An analysis of the S-matrix is most conveniently done in terms of charge carrying local field operators. In four-dimensional space-time, these comprise an algebra \mathcal{F} which contains the observables \mathcal{A} as the sub-algebra of invariants under a global gauge symmetry. However such a frame, the so-called Wick-Wightman-Wigner scenario, does not exist in the presence of non-Abelian braid group statistics in $d = 3$. As a unified minimal framework for both cases (permutation and braid group statistics), we shall use the field bundle formalism [9, 13, 17], which is quite rudimentary but suffices for the present purpose.

Particles and topological charges. We consider a collection Δ of superselection sectors, namely, representations of the observable algebra. To each $\pi \in \Delta$ there is associated a Hilbert space \mathcal{H}_π describing the corresponding charged states. Each \mathcal{H}_π carries a unitary continuous representation U_π of the translation group \mathbb{R}^d satisfying the relativistic spectrum condition (positivity of the energy). (In the proof of Thm. 1 we shall also use Lorentz covariance, but only in order not to burden the hypothesis.) The Hilbert space of the vacuum representation π_0 contains a unique (up to a factor) translation invariant vector Ω . We are particularly interested in so-called *massive single particle sectors*, describing the states of massive particles carrying topological charges. These are sectors π whose energy momentum spectrum has an isolated mass shell as the lower boundary.⁵ The charge described by π is then localizable with respect to the vacuum in space-like cones [9]. Namely, the representation π is equivalent to the vacuum representation when restricted to

⁵We thus assume implicitly that the vacuum representation does not contain massless particles.

the causal complement of any space-like cone. Within the set Δ of sectors one has intrinsic notions of charge composition $\pi_1 \times \pi_2$, and of charge conjugation, namely, every charge $\pi \in \Delta$ has a conjugate charge $\bar{\pi} \in \Delta$, characterized by the fact that the composition $\pi \times \bar{\pi}$ contains the vacuum representation. If π is a massive single particle representation, then so is $\bar{\pi}$, and it describes the corresponding anti-particles which have the same mass [16] and spin [8, 32]. There is also an intrinsic notion of exchange statistics relating $\pi \times \sigma$ and $\sigma \times \pi$. Namely, for every pair of localized morphisms π, σ in Δ there is a unitary operator

$$\varepsilon_{\pi\sigma} : \mathcal{H}_{\pi \times \sigma} \rightarrow \mathcal{H}_{\sigma \times \pi}, \quad (6)$$

the so-called statistics operator. The family of statistics operators satisfies the braid relations [18, Eq. (2.3)] and in four-dimensional space-time furnishes a representation of the permutation group due to the fact that here the monodromy operators $(\varepsilon_{\pi\pi})^2$ are trivial. In three-dimensional space-time, there may be non-trivial monodromy operators in which case the statistics operators furnish a representation of the braid group. Then one speaks of Plektons or, if the corresponding representation of the braid group is Abelian, of Anyons. Associated with each irreducible sector π is the statistics parameter λ_π and the statistics phase ω_π , defined by the relations

$$\varphi_\pi(\varepsilon_{\pi\pi}) = \lambda_\pi 1, \quad \omega_\pi = \frac{\lambda_\pi}{|\lambda_\pi|}, \quad (7)$$

where φ_π is a left inverse for π [21]. The statistics parameter is non-real precisely in the case of braid group statistics. In the Abelian case, $\varepsilon_{\pi\pi}$ is already trivial, namely, ω_π times the unit operator.

Charge carrying fields. For every $\pi \in \Delta$ there is a Banach space \mathcal{F}_π of charge carrying field operators, which act on the hermitean vector bundle $\mathcal{H} \doteq \bigcup_{\sigma \in \Delta} \mathcal{H}_\sigma$ in such a way that for every $\sigma \in \Delta$

$$\mathcal{F}_\pi : \mathcal{H}_\sigma \rightarrow \mathcal{H}_{\sigma \times \pi} \quad (8)$$

acts as a space of linear operators. Fields are localizable to the same extent to which the charges are localizable which they carry, namely in space-like cones. In the presence of braid group statistics in $d = 3$, the fields can have definite space-like commutation relations only if they carry some supplementary information in addition to the localization region. The possibility we choose is to consider paths in the set of space-like cones.⁶ (The following considerations are only relevant in $d = 3$.) Let H be the manifold of space-like directions,

$$H := \{e \in \mathbb{R}^3, e \cdot e = -1\}. \quad (9)$$

Every space-like cone C is of the form $C = a + \mathbb{R}^+ C^H$, where a is the apex of C and $C^H \doteq (C - a) \cap H$ is the set of space-like directions contained in C . We now look upon the space-like cone C as the union of all “strings” contained in it, $a + \mathbb{R}^+ e$,

⁶Two other possibilities are: To introduce a reference space-like cone from which all allowed localization cones have to keep space-like separated (this cone playing the role of a “cut” in the context of multi-valued functions) [9]; or a cohomology theory of nets of operator algebras as introduced by Roberts [37–39].

$e \in C^H$, and further identify such string with the pair $(a, e) \in \mathbb{R}^3 \times H$, quite in the sense of “string-localized quantum fields” [28,35]. Then C is identified with a subset of $\mathbb{R}^3 \times H$:

$$C \leftrightarrow \{a\} \times C^H, \quad (10)$$

where C^H is a double cone in H [33]. The region C^H is simply connected, whereas H itself has fundamental group \mathbb{Z} . Thus the portion of the universal covering space of $\mathbb{R}^3 \times H$ over a region $C \cong \{a\} \times C^H$ consists of a countable infinity of copies (“sheets”) of C . As usually, we identify the universal covering space of H with homotopy classes of paths in H starting at some fixed reference direction (the “base point” in H). We shall generically denote a sheet over C by \tilde{C} and call it a “path ending at C ”. (It is canonically homeomorphic to C , but contains in addition the information of a winding number distinguishing it from the other sheets over C .) These paths serve to label the localization regions of charged fields. Namely, for each path \tilde{C} there is a linear subspace $\mathcal{F}_\pi(\tilde{C})$ of \mathcal{F}_π , called the fields carrying charge π localized in \tilde{C} . This family is isotonous in the sense that

$$\mathcal{F}_\pi(\tilde{C}_1) \subset \mathcal{F}_\pi(\tilde{C}_2) \quad \text{if} \quad \tilde{C}_1 \subset \tilde{C}_2. \quad (11)$$

(We say that $\tilde{C}_1 \subset \tilde{C}_2$ if $C_1 \subset C_2$ and the corresponding paths \tilde{C}_1, \tilde{C}_2 differ by a path in C_2 .) The vacuum Ω has the Reeh-Schlieder property for the fields, i.e. for any path of space-like cones \tilde{C} there holds

$$(\mathcal{F}_\pi(\tilde{C})\Omega)^- = \mathcal{H}_\pi, \quad (12)$$

where the bar denotes the closure.

(Braid group) statistics. The statistics operators determine the commutation relations of causally separated fields, as follows. Let $d\theta$ be the angle one-form in some fixed Lorentz frame, and for a path \tilde{C} let $\theta(\tilde{C})$ be the set of corresponding “accumulated angles”, namely the interval

$$\theta(\tilde{C}) \doteq \left\{ \int_{\tilde{e}} d\theta : \tilde{e} \in \tilde{C}^H \right\} \quad (13)$$

on the real line. Given two paths \tilde{C}_1, \tilde{C}_2 with C_1 causally separate from C_2 , define the *relative winding number* $N(\tilde{C}_1, \tilde{C}_2)$ of \tilde{C}_2 w.r.t. \tilde{C}_1 to be the unique integer n such that

$$\theta(\tilde{C}_2) + 2\pi n < \theta(\tilde{C}_1) < \theta(\tilde{C}_2) + 2\pi(n+1).$$

(Note that this number n is independent of the Lorentz frame in which $d\theta$ is defined, and of the choice of reference direction.) Then for every $F_1 \in \mathcal{F}_{\pi_1}(\tilde{C}_1)$ and $F_2 \in \mathcal{F}_{\pi_2}(\tilde{C}_2)$ there holds the commutation relation

$$F_1 F_2 = \varepsilon_{\pi_1 \pi_2}(\tilde{C}_1, \tilde{C}_2) F_2 F_1, \quad (14)$$

where⁷

$$\varepsilon_{\pi_1 \pi_2}(\tilde{C}_1, \tilde{C}_2) = (\varepsilon_{\pi_1 \pi_2} \varepsilon_{\pi_2 \pi_1})^n \varepsilon_{\pi_1 \pi_2}, \quad n = N(\tilde{C}_1, \tilde{C}_2). \quad (15)$$

⁷This follows from Prop. 5.9 in [17], together with the fact that $\pi_0(V_{\varrho_2}) = 1$ and $\pi_0 \varrho_1(V_{\varrho_2}) = \pi_0(\varepsilon_{\varrho_2 \varrho_1} \varepsilon_{\varrho_1 \varrho_2})$, see [17, Eq. (5.2.3)].

We shall use below only two special cases, namely winding number 0 and -1 . $N(\tilde{C}_1, \tilde{C}_2) = 0$ means that for $\tilde{e}_i \in \tilde{C}_i^H$ the path $\tilde{e}_1 * \tilde{e}_2^{-1}$ is homotopic to a path which goes “directly” from C_2 to C_1 in the mathematically positive sense. (Figure 1 shows an example of this situation.) In this case $\varepsilon_{\pi_1 \pi_2}(\tilde{C}_1, \tilde{C}_2) = \varepsilon_{\pi_1 \pi_2}$. $N(\tilde{C}_1, \tilde{C}_2) = -1$ means that the paths $\tilde{e}_1 * \tilde{e}_2^{-1}$ are homotopic to paths which go directly from C_2 to C_1 in the mathematically *negative* sense (i.e., the roles of \tilde{C}_1 and \tilde{C}_2 in Fig. 1 are interchanged), and in this case $\varepsilon_{\pi_1 \pi_2}(\tilde{C}_1, \tilde{C}_2) = \varepsilon_{\pi_2 \pi_1}^{-1}$. (Of course, this is a consequence of the case $N(\tilde{C}_1, \tilde{C}_2) = 0$ if Eq. (14) is to be consistent.)

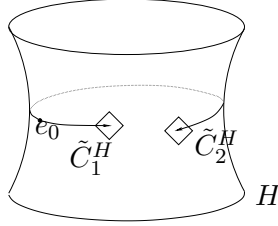


Figure 1: The relative winding number $N(\tilde{C}_1, \tilde{C}_2)$ of \tilde{C}_2 w.r.t. \tilde{C}_1 is zero. (e_0 is the base point where the paths start.)

Translation covariance. There is an action $x \mapsto \alpha_x$ of the translation group \mathbb{R}^d on the fields, under which these are covariant: Namely, for all $\pi \in \Delta$ and all localization paths \tilde{C} there holds

$$\alpha_x : \mathcal{F}_\pi(\tilde{C}) \rightarrow \mathcal{F}_\pi(x + \tilde{C}). \quad (16)$$

(Using the identification (10), the translation x does not act on the space-like directions C^H contained in C and hence not on \tilde{C}^H . It only translates the apex.) It is determined by the unitary representations U_π acting on the Hilbert spaces \mathcal{H}_π via

$$\alpha_x(F)\psi = U_{\sigma \times \pi}(x)FU_\sigma(-x)\psi, \quad F \in \mathcal{F}_\pi, \psi \in \mathcal{H}_\sigma. \quad (17)$$

Scattering states. Haag-Ruelle scattering theory has been developed in [22, 25], adapted to the setting of algebraic quantum field theory in [9, 12], and to theories with braid group statistics in [17]. This theory associates to n single particle states $\psi_k \in \mathcal{H}_{\pi_k}$ an outgoing and an incoming scattering state

$$\psi_1^{\text{out}} \times \cdots \times \psi_n^{\text{out}}, \quad \psi_1^{\text{in}} \times \cdots \times \psi_n^{\text{in}}$$

living in the Hilbert space of the composite sector $\pi_1 \times \cdots \times \pi_n$. In the case of permutation group statistics, these scattering states depend only on the single particle states ψ_k , as indicated by the notation. However, in the case of braid group statistics they also depend on the localization regions \tilde{C}_k in which the charged states ψ_k have been created from the vacuum. Since our argument relies crucially on this fact, we recall the construction.

First one constructs quasi-local creation operators as follows. Let $\pi \in \Delta$ be a massive single particle sector with mass m and let $F \in \mathcal{F}_\pi(\tilde{C})$ be a field operator which creates from the vacuum a single particle state with non-zero probability, i.e., the spectral support of $F\Omega$ has non-vanishing intersection with the mass hyperboloid

$H_m^+ \doteq \{p : p^2 = m^2, p_0 > 0\}$. Further, let $f \in \mathcal{S}(\mathbb{R}^4)$ be a Schwartz function whose Fourier transform⁸ \hat{f} has compact support contained in the open forward light cone intersecting the energy momentum spectrum only in the mass shell H_m^+ . If this intersection is contained in some $V \subset H_m^+$, namely if

$$\text{supp } \hat{f} \cap \text{spec } P_\pi \subset H_m^+, \quad \text{supp } \hat{f} \cap H_m^+ \subset V,$$

we shall say that f has *momentum support* in V . We shall use a fixed reference frame $u \equiv (1, \mathbf{0})$ and consider the set of velocities corresponding to V in this frame,

$$\Gamma(V) \doteq \left\{ \left(1, \frac{\mathbf{p}}{\omega(\mathbf{p})} \right), p = (p_0, \mathbf{p}) \in V \right\}, \quad (18)$$

where $\omega(\mathbf{p}) \doteq \sqrt{\mathbf{p}^2 + m^2}$. For $t \in \mathbb{R}$, let the function f_t be defined by multiplying the Fourier transform with the factor $e^{i(p_0 - \omega(\mathbf{p}))t}$. For large $|t|$, its support is essentially contained in the region $t\Gamma(V)$. More precisely [4, 22], for any $\varepsilon > 0$ there is a Schwartz function f_t^ε with support in $t\Gamma(V)^\varepsilon$, where Γ^ε denotes an ε -neighborhood of Γ , such that $f_t - f_t^\varepsilon$ converges to zero in the Schwartz topology for $|t| \rightarrow \infty$. Consider now the quasi-local operator

$$F(f_t) \doteq \int d^4x f_t(x) F(x),$$

where $F(x) \doteq \alpha_x(F)$. For large $|t|$, this operator is essentially localized in $\tilde{C} + t\Gamma(V)$. Namely, for any $\varepsilon > 0$, it can be approximated by the operator

$$F(f_t^\varepsilon) \in \mathcal{F}_\pi(\tilde{C} + t\Gamma(V)^\varepsilon) \quad (19)$$

in the sense that $\|F(f_t^\varepsilon) - F(f_t)\|$ is of fast decrease in t . This operator creates from the vacuum a single particle vector $F(f_t)\Omega = \hat{f}(P_\pi)F\Omega$, independent of t . (P_π denotes the generator of the translations $U_\pi(x)$.)

Consider now n massive single particle sectors $\pi_k, k = 1, \dots, n$, with mass m .⁹ To construct an outgoing scattering state from n corresponding single particle vectors, pick n localization regions \tilde{C}_k and compact sets V_k on the mass shell, such that the regions $\overline{C}_k + t\Gamma(V_k)$ are mutually space-like separated for large t .¹⁰ We then say that C_1, \dots, C_n are *future-admissible* for $V_1 \times \dots \times V_n$. Next, choose $F_k \in \mathcal{F}_{\pi_k}(\tilde{C}_k)$, and Schwartz functions f_k with momentum supports in V_k . Then the standard lemma of scattering theory, adapted to braid group statistics [17], asserts the following: The limit

$$\lim_{t \rightarrow \infty} F_1(f_{1,t}) \cdots F_n(f_{n,t}) \Omega \doteq (\psi_1, \tilde{C}_1) \overset{\text{out}}{\times} \cdots \overset{\text{out}}{\times} (\psi_n, \tilde{C}_n) \quad (20)$$

exists and depends only on the single particle vectors $\psi_k \doteq F_k(f_k)\Omega$ and, in the case of braid group statistics, on the localization regions \tilde{C}_k . Similarly, we say that C_1, \dots, C_n are *past-admissible* for $V_1 \times \dots \times V_n$ if the regions $\overline{C}_k + t\Gamma(V_k)$ are mutually causally separated for large $|t|$ with $t < 0$. In this case, the incoming scattering state

$$\lim_{t \rightarrow -\infty} F_1(f_{1,t}) \cdots F_n(f_{n,t}) \Omega \doteq (\psi_1, \tilde{C}_1) \overset{\text{in}}{\times} \cdots \overset{\text{in}}{\times} (\psi_n, \tilde{C}_n) \quad (21)$$

⁸The Fourier transform of a test function f is here $\hat{f}(p) \doteq \int d^4x f(x) e^{ip \cdot x}$.

⁹For notational simplicity we consider coinciding masses.

¹⁰ \overline{C} denotes the closure of C . Taking the closure makes sure that for suitable open neighborhoods Γ_k^ε of $\Gamma(V_k)$ in \mathbb{R}^d the regions $C_k + t\Gamma_k^\varepsilon$ are still mutually space-like separated for large t .

exists, and the above statements also hold. The limit vectors depend (linearly and) continuously on the single particle states, as a consequence of the following fact:

Lemma 4 (Scalar product of scattering states [17].) *Let ϕ_k be single particle vectors as above with the same charges π_k and spectral supports as ψ_k , $k = 1, \dots, n$. Let further T be a self-intertwiner of $\pi_1 \times \dots \times \pi_n$. Then there holds*

$$\begin{aligned} & ((\phi_1, \tilde{C}_1) \overset{\text{in}}{\times} \dots \overset{\text{in}}{\times} (\phi_n, \tilde{C}_n), T(\psi_1, \tilde{C}_1) \overset{\text{in}}{\times} \dots \overset{\text{in}}{\times} (\psi_n, \tilde{C}_n)) \\ & \qquad \qquad \qquad = \chi_1 \dots \chi_n(T) \prod_{k=1}^n (\phi_k, \psi_k), \end{aligned} \quad (22)$$

where χ_k is the standard right inverse of π_k , $k = 1, \dots, n$.

Note that by the Reeh-Schlieder property (12), every single-particle state $\psi \in \mathcal{H}_\pi$ with spectral support in some given $V \subset H_m^+$ can be approximated by a sequence of the form $F_\nu(f)\Omega$, where the field operators F_ν are localized in some fixed (arbitrary) region \tilde{C} . This allows the construction, by continuous extension, of scattering states

$$(\psi_1, \tilde{C}_1) \overset{\text{out}}{\times} \dots \overset{\text{out}}{\times} (\psi_n, \tilde{C}_n) \quad (23)$$

for any single particle vectors ψ_1, \dots, ψ_n with compact mutually disjoint spectral supports and localization regions \tilde{C}_k which are future-admissible for the supports of the ψ_k . The scattering states inherit the commutation relations of the fields, for example [17]

$$(\psi_1, \tilde{C}_1) \overset{\text{out}}{\times} (\psi_2, \tilde{C}_2) = \varepsilon_{\pi_1 \pi_2}(\tilde{C}_1, \tilde{C}_2) (\psi_2, \tilde{C}_2) \overset{\text{out}}{\times} (\psi_1, \tilde{C}_1). \quad (24)$$

The dependence of the scattering states on the space-time localization regions is well-known [17]. But since our argument relies on an explicit formula in the case $n = 2$, we shall exhibit this formula, together with the proof.

Lemma 5 (Change of localization regions) *Let ψ_1, ψ_2 be single particle states with spectral supports in V_1, V_2 , respectively. Suppose that all the pairs (K_1, K_2) , (K_1, C_2) , and (C_1, C_2) are future-admissible for $V_1 \times V_2$. Then*

$$(\psi_1, \tilde{K}_1) \overset{\text{out}}{\times} (\psi_2, \tilde{K}_2) = (\psi_1, \tilde{K}_1) \overset{\text{out}}{\times} (\psi_2, \tilde{C}_2) \quad (25)$$

$$= \varepsilon_{\pi_1 \pi_2}(\tilde{K}_1, \tilde{C}_2) \varepsilon_{\pi_2 \pi_1}(\tilde{C}_2, \tilde{C}_1) (\psi_1, \tilde{C}_1) \overset{\text{out}}{\times} (\psi_2, \tilde{C}_2). \quad (26)$$

The same holds for the respective incoming scattering states if (K_1, K_2) , (K_1, C_2) , and (C_1, C_2) are past-admissible for $V_1 \times V_2$.

Proof. In a first step, suppose that the vectors ψ_i are at the same time of the form $\psi_i = F_i(f_i)\Omega$ and $\psi_i = G_i(g_i)\Omega$, with $F_i \in \mathcal{F}_{\pi_i}(\tilde{C}_i)$ and $G_i \in \mathcal{F}_{\pi_i}(\tilde{K}_i)$, $i = 1, 2$. Then Eq. (25) holds obviously. In order to describe the dependence on the first localization region \tilde{K}_1 , we use Eq. (24) to commute the orders, then use Eq. (25) to replace \tilde{K}_1 by \tilde{C}_1 , and commute back:

$$\begin{aligned} (\psi_1, \tilde{K}_1) \overset{\text{out}}{\times} (\psi_2, \tilde{C}_2) &= \varepsilon_{\pi_1 \pi_2}(\tilde{K}_1, \tilde{C}_2) (\psi_2, \tilde{C}_2) \overset{\text{out}}{\times} (\psi_1, \tilde{K}_1) \\ &= \varepsilon_{\pi_1 \pi_2}(\tilde{K}_1, \tilde{C}_2) (\psi_2, \tilde{C}_2) \overset{\text{out}}{\times} (\psi_1, \tilde{C}_1) \\ &= \varepsilon_{\pi_1 \pi_2}(\tilde{K}_1, \tilde{C}_2) \varepsilon_{\pi_2 \pi_1}(\tilde{C}_2, \tilde{C}_1) (\psi_1, \tilde{C}_1) \overset{\text{out}}{\times} (\psi_2, \tilde{C}_2). \end{aligned}$$

Thus Eq. (26) holds in this special case. Now in general the ψ_i will not be exactly of the form $\psi_i = F_i(f_i)\Omega = G_i(g_i)\Omega$, but they can be approximated by such vectors. Then the above equations (25) and (26) hold by continuity. \square

We shall henceforth call a space-like cone C_1 *future- (past-) admissible* for a pair $V_1 \times V_2$ of compact sets on the mass shell if the regions $\overline{C_1} + t\Gamma(V_1)$ and $t\Gamma(V_2)$ are space-like separated for large positive (negative) t . (For then there exists a cone C_2 such that C_1, C_2 are admissible for $V_1 \times V_2$ in the sense of the earlier definition.) Since the scattering state is independent of the localization region C_2 of ψ_2 , we shall write $(\psi_1, \tilde{C}_1) \overset{\text{out}}{\times} \psi_2$ instead of $(\psi_1, \tilde{C}_1) \overset{\text{out}}{\times} (\psi_2, \tilde{C}_2)$.

We shall consider *special space-like cones* in the sense of Bros and Epstein [5], referring to the reference frame u and its rest space $\Sigma \doteq u^\perp$. Let \underline{C} be an open, salient cone in Σ with apex at the origin. Then its causal completion $C = \underline{C}''$ is a special space-like cone. We shall make use of the following simple observations.

Lemma 6 *i) Let C_1, C_2 be space-like cones with apices at the origin, and let Γ_1, Γ_2 be compact disjoint sets in velocity space. If the condition " $\overline{C_1} + t\Gamma_1$ and $\overline{C_2} + t\Gamma_2$ are causally separated" holds for some $t > 0$ then it holds for all $t > 0$. (The same is true for negative t .)*

ii) Let C_1, C_2 be special space-like cones and let $p_1 \neq p_2 \in H_m^+$. C_1, C_2 are future- and past- admissible for (p_1, p_2) if, and only if, the span of $\{p_1, p_2\}$ has trivial intersection with the closure of the difference cone $C \doteq C_2 - C_1$,

$$\text{span}\{p_1, p_2\} \cap \overline{C} = \{0\}. \quad (27)$$

Proof. Let $t, t' > 0$ with $t' = \lambda t$. $\overline{C_1} + t\Gamma_1$ and $\overline{C_2} + t\Gamma_2$ are causally separated if and only if

$$0 < (x + t(v_2 - v_1))^2 \equiv \lambda^{-2} (\lambda x + t'(v_2 - v_1))^2$$

for all $v_i \in \Gamma_i$ and $x \in C$, where $C \doteq C_2 - C_1$. If this condition holds for t then it also holds for t' since $x \in C$ iff $\lambda x \in C$, $\lambda > 0$. This proves *i)*. To prove *ii)*, let v_i be the velocities in the given reference frame corresponding to p_i , namely

$$v_i \doteq (1, \frac{\mathbf{p}_i}{\omega(\mathbf{p}_i)}). \quad (28)$$

Note that for special space-like cones, $C_i = (\underline{C_i})''$, $\overline{C_1} + tv_1$ is causally separated from $\overline{C_2} + tv_2$ if and only if $\underline{C_1} + tv_1$ is disjoint from $\underline{C_2} + tv_2$. Thus, by part *i)*, C_i are future and past-admissible for (p_1, p_2) iff $\underline{C_1} + tv_1$ is disjoint from $\underline{C_2} + tv_2$ for all $t > 0$ and for all $t < 0$, that is, iff

$$\mathbb{R}(v_1 - v_2) \cap \overline{C} = \{0\}. \quad (29)$$

Since \underline{C} is contained in $\Sigma = (1, \mathbf{0})^\perp$, and the span of v_1, v_2 is a time-like hyper-plane whose intersection with Σ is just $\mathbb{R}(v_1 - v_2)$, the above equation is equivalent with

$$\{0\} = \text{span}\{v_1, v_2\} \cap \overline{C} = \text{span}\{v_1, v_2\} \cap \overline{C}.$$

But the span of v_1, v_2 obviously coincides with the span of p_1, p_2 , and the proof is complete. \square

3 Two-Particle LSZ Formulae

The LSZ-relations, which relate S-matrix elements with time-ordered products of fields, have been derived by Hepp within Haag-Ruelle theory [22], see [10] for a review. Using the arguments of [10], we verify here that certain LSZ-relations are valid in the case of localization in space-like cones and of braid group statistics, under certain hypothesis on the space-time localization regions and momentum supports. Let $\pi_1, \pi_2 \in \Delta$, let C_1 and C_2 be causally separated special space-like cones and \tilde{C}_i be paths ending at C_i , and let

$$\varepsilon \doteq \varepsilon_{\pi_1 \pi_2}(\tilde{C}_1, \tilde{C}_2). \quad (30)$$

For $F_1 \in \mathcal{F}_{\pi_1}(\tilde{C}_1)$ and $F_2 \in \mathcal{F}_{\pi_2}(\tilde{C}_2)$ let $TF_1(x)F_2(y)$ be the time-ordered product:

$$TF_1(x)F_2(y) = \theta(x^0 - y^0)F_1(x)F_2(y) + \theta(y^0 - x^0)\varepsilon F_2(y)F_1(x). \quad (31)$$

(Here, θ is the Heaviside function, $\theta(t) = 0$ if $t < 0$ and $\theta(t) = 1$ if $t \geq 0$.) Let now V_1, V_2 be compact subsets of the mass shell such that C_1, C_2 is both future and past-admissible for $V_1 \times V_2$, let f_1, f_2 be test functions with momentum supports in V_1 and V_2 , respectively, and denote $g_{i,t}(x) \doteq f_{i,t}(x) - f_{i,-t}(x)$, $i = 1, 2$. Then for fixed $s > 0$ and sufficiently large $t > 0$ one has

$$\begin{aligned} & \int d^4x d^4y g_{1,s}(x) g_{2,t}(y) T(F_1(x)F_2(y))\Omega \\ & \equiv \int d^4x g_{1,s}(x) \int d^4y \{f_{2,t}(y) T(F_1(x)F_2(y)) - f_{2,-t}(y) T(F_1(x)F_2(y))\}\Omega \\ & \simeq \int d^4x g_{1,s}(x) \int d^4y \{f_{2,t}(y) \varepsilon F_2(y)F_1(x) - f_{2,-t}(y) F_1(x)F_2(y)\}\Omega \\ & = \varepsilon F_2(f_{2,t})F_1(g_{1,s})\Omega - F_1(g_{1,s})F_2(f_{2,t})\Omega \\ & = F_1(f_{1,-s})F_2(f_2)\Omega - F_1(f_{1,s})F_2(f_2)\Omega. \end{aligned} \quad (32)$$

In the third line (“ \simeq ” means equality up to terms which vanish in the limit $t \rightarrow \infty$) we have used the fact that the supports of $g_{1,s}$ and $f_{2,\pm t}$ are essentially contained in the regions $s\Gamma(V_1) \cup -s\Gamma(V_1)$ and $\pm t\Gamma(V_2)$, respectively, which are chronologically ordered as

$$-tv_2^0 < |sv_1^0| < tv_2^0, \quad v_i \in \Gamma(V_i),$$

for fixed s and sufficiently large $t > 0$. In the last equation we have used that $F_1(g_{1,s})\Omega \equiv F_1(f_{1,s})\Omega - F_1(f_{1,-s})\Omega = 0$ since $F_1(f_{1,s})\Omega$ is independent of s . The last expression in Eq. (32) converges for $s \rightarrow \infty$ to

$$F_1(f_1)\Omega \stackrel{\text{in}}{\times} F_2(f_2)\Omega - F_1(f_1)\Omega \stackrel{\text{out}}{\times} F_2(f_2)\Omega.$$

On the other hand, the limit of the first line in Eq. (32) can be calculated in Fourier space, noting that

$$\frac{\widehat{f_t}(p) - \widehat{f_{-t}}(p)}{p^2 - m^2} \rightarrow 2\pi i \delta_m(p) \hat{f}(p), \quad \delta_m(p) \doteq \delta(p^2 - m^2)\theta(p_0), \quad (33)$$

for $t \rightarrow \infty$. Let ϕ be a state vector in $\mathcal{H}_{\pi_1 \times \pi_2}$ with compact spectral support, and let $t(x_1, x_2)$ be the “time-ordered function”

$$t(x_1, x_2) \doteq (\phi, T(F_1(x_1)F_2(x_2))\Omega). \quad (34)$$

Define the amputated time-ordered function \check{t}_{amp} by

$$\check{t}_{\text{amp}}(p_1, p_2) \doteq (p_1^2 - m^2)(p_2^2 - m^2) \check{t}(p_1, p_2), \quad (35)$$

where \check{t} is the inverse Fourier transform of t .¹¹ Then the scalar product of the first line in Eq. (32) with ϕ is

$$t(g_{1,s} \otimes g_{2,t}) \equiv \check{t}_{\text{amp}}\left(\frac{\hat{g}_{1,s}}{p_1^2 - m^2} \otimes \frac{\hat{g}_{2,t}}{p_2^2 - m^2}\right)$$

and the limit in s, t is *formally*, by Eq. (33),

$$(2\pi i)^2 (\delta_m \otimes \delta_m \cdot \check{t}_{\text{amp}})(\hat{f}_1 \otimes \hat{f}_2) \equiv (2\pi i)^2 \int_{H_m^+ \times H_m^+} d\mu(p_1, p_2) \hat{f}_1(p_1) \hat{f}_2(p_2) \check{t}_{\text{amp}}(p_1, p_2).$$

As we see in Lemma 7 below, this can be justified under the stated hypothesis on C_i and V_i . We then conclude from Eq. (32):

$$\begin{aligned} (2\pi i)^2 \int_{H_m^+ \times H_m^+} d\mu(p_1, p_2) \hat{f}_1(p_1) \hat{f}_2(p_2) \check{t}_{\text{amp}}(p_1, p_2) |_{H_m^+ \times H_m^+} \\ = (\phi, F_1(f_1)\Omega \times^{\text{in}} F_2(f_2)\Omega - F_1(f_1)\Omega \times^{\text{out}} F_2(f_2)\Omega). \end{aligned} \quad (36)$$

This is the relevant LSZ relation. (See [1, Cor. 5.10] and [10] in the case of compact localization.)

It remains to show that \check{t}_{amp} can be multiplied with the distribution $\delta_m \otimes \delta_m$ or, equivalently, can be restricted to the two-fold mass shell

$$H_m^2 \doteq H_m^+ \times H_m^+.$$

To this end it suffices to show that, over H_m^2 , the wave front set of \check{t}_{amp} is disjoint from the co-normal bundle of H_m^2 (in which the wave front set of $\delta_m \otimes \delta_m$ is contained), see [23, Thm. 8.2.10]. In order to determine the wave front set, we use a standard argument involving the “two-point”, commutator, advanced and retarded functions

$$\begin{aligned} w(x_1, x_2) &\doteq (\phi, F_1(x_1)F_2(x_2)\Omega), \\ c(x_1, x_2) &\doteq (\phi, F_1(x_1)F_2(x_2) - \varepsilon F_2(x_2)F_1(x_1)\Omega), \\ a(x_1, x_2) &\doteq \theta(x_1^0 - x_2^0) c(x_1, x_2), \\ r(x_1, x_2) &\doteq -\theta(x_2^0 - x_1^0) c(x_1, x_2). \end{aligned}$$

Obviously $c = a - r$ and $t = r + w$. The amputated two-point, advanced and retarded functions \check{w}_{amp} , \check{a}_{amp} , \check{r}_{amp} are defined analogously as \check{t}_{amp} . Now the inverse Fourier transform of w is a measure in the second variable which has support in the energy momentum spectrum, namely

$$\check{w}(\hat{f}_1 \otimes \hat{f}_2) = \int \hat{f}_2(p) d(\phi, F_1(f_1)E_p F_2\Omega),$$

where dE_p is the energy-momentum spectral measure. Hence by the mass gap condition, the amputated two-point function vanishes in a neighborhood of the mass

¹¹The inverse Fourier transform of a distribution t is given by $\check{t}(\hat{f}) = t(f)$.

shell (which contains no off-shell spectral points). Since $t \equiv r + w$, this implies that the amputated time-ordered function \check{t}_{amp} coincides in that neighborhood with the amputated retarded function \check{r}_{amp} . By the same token, the amputated commutator function vanishes on a neighborhood of H_m^2 , which implies that the amputated advanced and retarded functions coincide on that neighborhood. Thus,

$$\check{t}_{\text{amp}} = \check{a}_{\text{amp}} = \check{r}_{\text{amp}} \quad \text{on a neighborhood of } H_m^2. \quad (37)$$

Hence the wave front sets over H_m^2 of these distributions coincide. Now the wave front sets of \check{a}_{amp} , \check{r}_{amp} are related with the supports of a , r . We shall use total and relative momenta (P, p) as well as the center-of-mass and relative variables (X, x) ¹²

$$P \doteq p_1 + p_2, \quad p \doteq \frac{1}{2}(p_1 - p_2), \quad X \doteq \frac{1}{2}(x_1 + x_2), \quad x \doteq x_1 - x_2. \quad (38)$$

By locality, the commutator function vanishes if $x \in C'$, where

$$C \doteq C_2 - C_1, \quad (39)$$

i.e., it has support in $(X, x) \in \mathbb{R}^4 \times \overline{C + (V_+ \cup V_-)}$. Therefore the advanced and retarded functions have their supports in the closed cones $\mathbb{R}^4 \times \overline{C + V_{\pm}}$, respectively. The same holds of course after applying the differential operator $(\square_1 + m^2)(\square_2 + m^2)$. Therefore the wave front sets of the amputated advanced and retarded products are contained in the same cones, see [23, Lemma 8.1.7] and [36, Thm. IX.44]. Over the neighborhood where \check{t}_{amp} coincides with the amputated retarded and advanced products, the wave front set of \check{t}_{amp} is thus contained in the intersection of these cones, which is

$$\mathbb{R}^4 \times \overline{C}. \quad (40)$$

Now the co-normal bundle of H_m^2 over a point $(p_1, p_2) \in H_m^2$ is, in terms of the corresponding total and relative momentum (P, p) ,

$$(T_{(P,p)} H_m^2)^{\perp} = \mathbb{R}^4 \times \text{span}\{P, p\}. \quad (41)$$

It has trivial intersection with the wave front set (40) over (P, p) if \overline{C} has trivial intersection with $\text{span}\{P, p\} \equiv \text{span}\{p_1, p_2\}$. Part *ii*) of Lemma 6 now implies:

Lemma 7 (Mass shell restriction of \check{t}_{amp} .) *If C_1 and C_2 are special space-like cones such that C_1, C_2 is future- and past- admissible for $V_1 \times V_2 \subset H_m^2$, then \check{t}_{amp} can be restricted to $V_1 \times V_2$.*

We have now verified that the LSZ-relation (36) is valid in the case of localization in space-like cones and of braid group statistics, given that the localization regions of F_1 and F_2 are causally separated and are future- and past- admissible for the momentum supports of f_1 and f_2 .

¹²Recall that $P \cdot X + p \cdot x = p_1 \cdot x_1 + p_2 \cdot x_2$, hence the Fourier transform intertwines the respective variable transformations.

4 Analyticity of the Amputated Time-Ordered Function

The next step is to show that the amputated time-ordered function on the left hand side of Eq. (36) is the boundary value of an analytic function in the relative momentum. We adapt the standard arguments to verify that this holds also in the case of space-like cone localization and of braid group statistics.

Let \tilde{C}_i and let $F_i \in \mathcal{F}_{\pi_i}(\tilde{C}_i)$ be as in the last section, $i = 1, 2$, and suppose that the difference cone $C \doteq C_2 - C_1$ is salient (which is the case if $C_2 = -C_1$). Recall from above that the corresponding advanced and retarded functions have their supports in the closed cones $(X, x) \in \mathbb{R}^4 \times \overline{C + V_\pm}$, respectively. It follows [36, Thm. IX.16] that the inverse Fourier transforms $\check{a}(P, p)$, $\check{r}(P, p)$ are, in the p -variable, boundary values of two functions which are analytic in $\mathbb{R}^4 + i(V_\pm \cap C^*)$ respectively, where C^* is the dual cone, namely the set of all $q \in \mathbb{R}^4$ such that $q \cdot x > 0$ for all x in $\overline{C} \setminus \{0\}$. The same holds of course for the amputated advanced and retarded functions. But these coincide by Eq. (37) on a neighborhood of the two-fold mass shell H_m^2 . Recalling that H_m^2 corresponds under the transformation (38) to the set of (P, p) such that $P^2 \geq 4m^4$, $p \cdot P = 0$ and $p^2 + \frac{1}{4}P^2 = m^2$, this means that \check{a}_{amp} and \check{r}_{amp} coincide for fixed P with $P^2 \geq 4m^4$ on a neighborhood U of the space-like two-sphere consisting of the on-shell relative momenta p for the given P ,

$$M_P \doteq \{p \in \mathbb{R}^4 : p \cdot P = 0, p^2 = m^2 - \frac{1}{4}P^2 < 0\}. \quad (42)$$

The edge-of-the-wedge theorem for oblique tubes [7, 14] then asserts that the distributions \check{a}_{amp} and \check{r}_{amp} in fact are boundary values of a single function which is analytic in an open set Θ_C (a “local tube”) which contains all points $p + iq$ such that $p \in U$ and q is contained in the convex hull of $V_+ \cap C^*$ and $V_- \cap C^*$, namely in

$$C^\dagger \doteq (V_+ \cap C^*) + (V_- \cap C^*), \quad (43)$$

and satisfies a certain bound in norm, $\|q\| < \rho(p)$. Since the amputated time-ordered function \check{t}_{amp} coincides with \check{a}_{amp} and \check{r}_{amp} on U , c.f. Eq. (37), it shares this analyticity property. Analyticity in Θ_C is, however, not sufficient for our later purpose: We shall wish to conclude from the vanishing of \check{t}_{amp} on a small open set in M_P that it vanishes on a larger set in M_P . We therefore have to consider the complexification of M_P . By the above considerations, we know that $\check{t}_{\text{amp}}(P, p)$ has, for fixed P , an analytic extension in p into the intersection of Θ_C with the complexification of M_P , which we denote by

$$\Theta_{P,C}.$$

$\check{t}_{\text{amp}}(P, p)$ is the boundary value of this analytic function for all $p \in M_P$ which lie at the real boundary of $\Theta_{P,C}$. Let us denote this set by

$$M_{P,C}.$$

Now as a consequence of the fact that C^\dagger is a salient cone which does not contain the origin, $M_{P,C}$ does not cover all of M_P . (This contrasts the case of compact localization.) However, depending on C , it can be made to cover a large part of M_P . The following proposition characterizes the the real boundary $M_{P,C}$ of the domain of analyticity of the amputated time-ordered function, see also [5, Lemma 5.1].

Proposition 8 (Analyticity domain of \check{t}_{amp} .) *i) An on-shell relative momentum $p \in M_P$ is in $M_{P,C}$ if, and only if, the span of $\{P, p\}$ has trivial intersection with the closure of C ,*

$$\text{span}\{P, p\} \cap \overline{C} = \{0\}. \quad (44)$$

ii) $M_{P,C}$ covers M_P up to two anti-podal neighborhoods $U_{P,C} \cup -U_{P,C}$, where $U_{P,C}$ is the intersection of M_P with the projection of \overline{C} onto P^\perp ,

$$U_{P,C} = M_P \cap E_P^\perp \overline{C}. \quad (45)$$

Note that by part *ii)* of Lemma 6 the condition (44) is satisfied for all $(p_1, p_2) \in V_1 \times V_2$ if C_1, C_2 is future- and past- admissible for $V_1 \times V_2$. Note also that in $d \geq 4$ the set $M_{P,C}$ is connected, whereas in $d = 3$ it has two connected components.

Proof. The complexification of M_P consists of all $p + iq \in \mathbb{C}^4$ with $p, q \in P^\perp$ and $p \cdot q = 0$, $p^2 - q^2 = m^2 - \frac{1}{4}P^2$. The real boundary of $\Theta_{P,C}$ therefore consists of all points $p \in M_P$ for which there is some $q \in P^\perp \cap C^\dagger$ orthogonal to p . (For then

$$\frac{\sqrt{\mu^2 - \lambda^2 q^2}}{\mu} p + i\lambda q$$

converges to p within Θ_C if $\lambda \rightarrow 0$. Here, $\mu^2 \doteq \frac{1}{4}P^2 - m^2$.) In other words, p belongs to the real boundary if, and only if,

$$\Pi^\perp \cap C^\dagger \neq \emptyset, \quad (46)$$

where Π is the time-like plane spanned by P, p . Since $C^\dagger \subset C^*$, it is obvious that (46) implies (44). To prove that (44) implies (46), namely the “if” statement of part *i)*, we show in a first step that the condition (44) allows for the construction of a wedge which contains $\overline{C} \setminus \{0\}$ and whose edge has non-trivial intersection with Π . The hypothesis (44) implies that the intersection of Π with the closure of the base \underline{C} of our special space-like cone is trivial. Thus, the intersection of Π with the space-like hyper-surface $\Sigma \supset \underline{C}$, which is a one-dimensional space-like subspace L , has trivial intersection with \overline{C} . Since \underline{C} is a salient cone in the $(d-1)$ -dimensional hyper-plane Σ , there exists a $(d-2)$ -dimensional linear hyper-plane E of Σ which contains the line L and still has trivial intersection with \overline{C} . Then E divides Σ into two connected components and \underline{C} is contained in one of them. Let W be the causal completion of this half-space of Σ . Then W is a wedge region¹³ which contains $\overline{C} \setminus \{0\}$ and whose edge, E , has non-trivial intersection with Π , namely, $\Pi \cap E = L$. Our second step is to show that the existence of such a wedge implies the claimed Eq. (46). The upper and lower boundaries of W contain two light-like rays $\mathbb{R}l_\pm \subset \partial V_\pm$. They lie in the orthogonal complement of E and characterize W as follows:

$$W = \{x \in \mathbb{R}^d : x \cdot l_+ < 0, x \cdot l_- < 0\}. \quad (47)$$

¹³A *wedge* is a region in Minkowski space which arises from the standard wedge

$$W_1 \doteq \{x \in \mathbb{R}^d : |x^0| < x^1\},$$

by a Poincaré transformation. Its *edge* is the intersection of the upper and lower bordering light-like half-planes.

Now by construction, Π and E span a $d - 1$ -dimensional time-like hyper-plane and $\Pi^\perp \cap E^\perp$ is a one-dimensional space-like subspace. Let q be in the component of this line which is contained in the opposite wedge $-W$. Then

$$q = -l'_+ - l'_- \in \Pi^\perp \cap E^\perp,$$

where $l'_\pm \in \partial V_\pm$ are suitable positive multiples of l_\pm . The vectors $-l'_\pm$ are in the dual of C by Eq. (47). Thus, the time-like vectors

$$q_\pm \doteq -l'_\mp \pm \varepsilon(l'_+ - l'_-) \in V_\pm,$$

are still contained in the dual of C for sufficiently small $\varepsilon > 0$. We have thus found $q \in \Pi^\perp$ which can be written as $q = q_+ + q_-$ with q_\pm in $V_\pm \cap C^* = C^\dagger$, i.e., Eq. (46) holds. This completes the proof of *i*). Now Eq. (44) is equivalent with

$$\mathbb{R}p \cap E_P^\perp \overline{C} = \{0\},$$

which proves *ii*). □

5 Proof of the Theorems

5.1 The four (and higher) dimensional case.

Recalling the situation of Theorem 1, suppose there are sets of incoming momenta $U_1, U_2 \subset H_m^+$ and of outgoing momenta $V_1, V_2 \subset H_m^+$, all mutually disjoint, cf. Eq. (1), and compatible with energy-momentum conservation: Namely, $U_1 + U_2$ and $V_1 + V_2$ are neighborhoods of some $P \in V_+$, $P^2 \geq 4m^2$. The hypothesis is that there is no elastic two-particle scattering from $U_1 \times U_2$ into $V_1 \times V_2$ in the channel $\pi_1 \times \pi_2 \rightarrow \pi_1 \times \pi_2$, namely:

$$\left(\phi_1 \overset{\text{in}}{\times} \phi_2, \psi_1 \overset{\text{out}}{\times} \psi_2 \right) = 0 \quad (48)$$

for all single-particle states ϕ_i in the sectors π_i with respective spectral supports in U_i , and all ψ_i in the same sectors π_i and with supports in V_i . Due to the assumed disjointness of the energy-momentum supports, the state $\psi_1 \overset{\text{in}}{\times} \psi_2$ has vanishing scalar product with $\phi_{\text{in}} \doteq \phi_1 \overset{\text{in}}{\times} \phi_2$, and the hypothesis can be rephrased as

$$\left(\phi_{\text{in}}, \psi_1 \overset{\text{in}}{\times} \psi_2 - \psi_1 \overset{\text{out}}{\times} \psi_2 \right) = 0 \quad (49)$$

for all ψ_i with spectral supports in V_i . We wish to show that Eq. (49) also holds if the spectral supports of the ψ_i are contained in other subsets V'_i of the mass shell, with ϕ_{in} fixed. We shall assume that the sets V_i and V'_i are small neighborhoods of 4 on-shell momenta $p_1 \neq p_2$ and $p'_1 \neq p'_2$, and that

$$V'_1 + V'_2 \subset V \doteq V_1 + V_2.$$

We then choose special space-like cones $C_1, C_2 \doteq -C_1$ such that the pair C_1, C_2 is future- and past- admissible *both* for $V_1 \times V_2$ and for $V'_1 \times V'_2$.

Lemma 9 *There exists such a special space-like cone C_1 .*

Proof. Let v_i and v'_i be the velocities corresponding to p_i and p'_i , respectively, as in Eq. (28). Choose a unit vector e in Σ which is disjoint from $\mathbb{R}(v_1 - v_2)$ and from $\mathbb{R}(v'_1 - v'_2)$, and let \underline{C}_1 be a cone in Σ centered at e , with sufficiently small opening angle such that \underline{C}_1 still has trivial intersection both with $\mathbb{R}(v_1 - v_2)$ and $\mathbb{R}(v'_1 - v'_2)$. Let C_1 be its causal completion and let $C_2 \doteq -C_1$. Then C_1, C_2 are future- and past-admissible for $\{(v_1, v_2)\}$ and for $\{(v'_1, v'_2)\}$, see Eq. (29) in the proof of Lemma 6. The same holds for the set $V_1 \times V_2$ and $V'_1 \times V'_2$ if these are small enough. \square

The difference cone $C \doteq C_2 - C_1$ is now just $C \equiv -C_1$. Let us denote by $M_{V,C}$ the set of pairs of on-shell momenta whose total momentum is contained in V and whose relative momenta are contained in the real boundary of the analyticity domain of the corresponding amputated time-ordered functions. By virtue of Proposition 8 this set is given by

$$M_{V,C} \doteq \{(p'_1, p'_2) \in H_m^+ \times H_m^+ : p'_1 + p'_2 \in V, \text{span}\{p'_1, p'_2\} \cap \overline{C} = \{0\}\}. \quad (50)$$

Now Eq. (49) holds for all ψ_i of the form $\psi_i = F_i(f_i)\Omega$, where $F_i \in \mathcal{F}_{\pi_i}(C_i)$ and where f_i are test functions with respective momentum supports in V_i . The LSZ relation (36) then implies that

$$\check{t}_{\text{amp}}(p_1, p_2) = 0 \quad (51)$$

for all $p_i \in V_i$. Here, \check{t}_{amp} is the amputated time-ordered function, with $t(x_1, x_2)$ being defined as in Eq. (34) with $\phi = \phi_{\text{in}}$ and F_i as above. Since our pair of cones C_1, C_2 is future- and past-admissible for $V_1 \times V_2$, the remark after Proposition 8 implies that the relative momenta corresponding to $V_1 \times V_2$ are, for all $P' \in V$, contained in the boundary $M_{P',C}$ of the domain of analyticity of $\check{t}_{\text{amp}}(P', \cdot)$. Now in four- (and higher) dimensional space-time, $M_{P',C}$ is connected and therefore Eq. (51) and analyticity of \check{t}_{amp} imply that $\check{t}_{\text{amp}}(P', p')$ vanishes for all $P' \in V$ and $p' \in M_{P',C}$, that is to say, \check{t}_{amp} vanishes on $M_{V,C}$. Since our cones C_1, C_2 are future- and past-admissible also for the set $V'_1 \times V'_2$, the latter is also contained in the set $M_{V,C}$ where we have just shown that \check{t}_{amp} vanishes. Again by the LSZ relation (36), we conclude that Eq. (49) holds for all ψ_i of the form $\psi_i = F_i(f_i)\Omega$, where $F_i \in \mathcal{F}_{\pi_i}(C_i)$ and the momentum supports of \hat{f}_i are contained in V'_i . By continuity of the scattering states and the Reeh-Schlieder property, we arrive at the following result:

Proposition 10 (Triviality of the S-matrix element in $d \geq 4$.) *Let V_i and U_i be as in Theorem 1, $i = 1, 2$, and let V'_1, V'_2 be disjoint, sufficiently small, subsets of the mass shell such that $V'_1 + V'_2 \subset V = V_1 + V_2$. Then there holds*

$$\left(\phi_1 \overset{\text{in}}{\times} \phi_2, \psi_1 \overset{\text{in}}{\times} \psi_2 \right) = \left(\phi_1 \overset{\text{in}}{\times} \phi_2, \psi_1 \overset{\text{out}}{\times} \psi_2 \right) \quad (52)$$

for all single-particle states ψ_i with spectral supports in V'_i and for all ϕ_i with spectral supports contained in the sets U_i , $i = 1, 2$.

By Lorentz covariance, the same holds if U_1 and U_2 are subject to a common Lorentz transformation. Since V was arbitrary in Theorem 1, it follows by linearity and continuity that Eq. (52) holds for all single-particle states ϕ_i, ψ_i . In other words,

we have shown that $E_{\text{in}}^{(2)} S E_{\text{in}}^{(2)} = E_{\text{in}}^{(2)}$, where $E_{\text{in}}^{(2)}$ is the projection onto the closed subspace of incoming two-particle scattering states, and S is the S-matrix,

$$S : \psi_1 \times^{\text{in}} \psi_2 \mapsto \psi_1 \times^{\text{out}} \psi_2.$$

But then the assumption that $S E_{\text{in}}^{(2)}$ has other components than in $E_{\text{in}}^{(2)} \mathcal{H}$ contradicts the isometry of the S-matrix. It then follows that $S E_{\text{in}}^{(2)} = E_{\text{in}}^{(2)}$, i.e. Eq. (4) holds. This concludes the proof of Theorem 1.

5.2 The three-dimensional case.

In three space-time dimensions two complications arise: Firstly, the scattering states depend on the space-time localization regions, and secondly, the set of analyticity $M_{V,C}$ has two connected components and the above argument only leads to the conclusion that \tilde{t}_{amp} vanishes on the component which contains $V_1 \times V_2$.

The hypothesis of Theorem 2 is: There are sets of incoming momenta $U_1, U_2 \subset H_m^+$ and of outgoing momenta $V_1, V_2 \subset H_m^+$, all mutually disjoint, cf. Eq. (1), and compatible with energy-momentum conservation, c.f. Eq. (1), and there is *some* path \tilde{C}_0 , future-admissible for $V_1 \times V_2$, such that

$$(\phi_1 \times^{\text{in}} \phi_2, (\psi_1, \tilde{C}_0) \times^{\text{out}} \psi_2) = 0 \quad (53)$$

holds for all single-particle states ϕ_i in the sectors π_i with respective spectral supports in U_i ,¹⁴ and all ψ_i in the same sectors π_i and with supports in V_i , $i = 1, 2$. We may assume that the spectral supports U_i are small enough as to satisfy

$$U_1 + U_2 \subset V \equiv V_1 + V_2. \quad (54)$$

Suppose that \tilde{C}_1 is a path ending at a special space-like cone C_1 such that

- (a) C_1 is future- and past- admissible both for $V_1 \times V_2$ and for $U_1 \times U_2$. (Then $V_1 \times V_2$ and $U_2 \times U_2$ are contained in $M_{V,C}$, $C \doteq -C_1$, by the remark after Prop. 8.)
- (b) $V_1 \times V_2$ and $U_1 \times U_2$ are in one and the same connected component of $M_{V,C}$.
- (c) The path \tilde{C}_1 is equivalent with \tilde{C}_0 for outgoing scattering states with momentum supports in $V_1 \times V_2$, namely,

$$(\psi_1, \tilde{C}_1) \times^{\text{out}} \psi_2 = (\psi_1, \tilde{C}_0) \times^{\text{out}} \psi_2 \quad (55)$$

holds for all ψ_i with spectral supports in V_i , $i = 1, 2$.

Due to (c), we may substitute \tilde{C}_1 for \tilde{C}_0 right at the beginning of our discussion, namely in Eq. (53), and as in the last section we conclude that

$$(\phi_{\text{in}}, (\psi_1, \tilde{C}_1) \times^{\text{in}} \psi_2 - (\psi_1, \tilde{C}_1) \times^{\text{out}} \psi_2) = 0 \quad (56)$$

holds for all ψ_i of the form $\psi_i = F_i(f_i)\Omega$, where $F_i \in \mathcal{F}_{\pi_i}(\tilde{C}_i)$ with $C_2 \doteq -C_1$ and \tilde{C}_2 arbitrary, and where the momentum supports of f_i are contained in U_i . Continuity and the Reeh-Schlieder property then imply:

¹⁴The space-time localization regions for the state $\phi_1 \times^{\text{in}} \phi_2$ will not play any significant role.

Proposition 11 (Local triviality of the S-matrix element in $d = 3$.)

Suppose the spectral supports U_i are small enough as to satisfy $U_1 + U_2 \subset V$. Then Eq. (56) holds for all single-particle states ψ_i with momentum supports in U_i , $i = 1, 2$, and for all paths \tilde{C}_1 with the above properties (a), (b), (c).

We now show that this result is incompatible with braid group statistics, the idea being as follows. We construct two paths \tilde{C}_1 and \tilde{K}_1 which both satisfy the properties (a) through (c), and such that for all ψ_i with spectral supports in U_i the outgoing scattering states are invariant under a change of localization regions from \tilde{C}_1 to \tilde{K}_1 , whereas the corresponding incoming scattering states differ by a monodromy operator ε_M , that is

$$(\psi_1, \tilde{C}_1) \times^{\text{out}} \psi_2 = (\psi_1, \tilde{K}_1) \times^{\text{out}} \psi_2, \quad (\psi_1, \tilde{C}_1) \times^{\text{in}} \psi_2 = \varepsilon_M (\psi_1, \tilde{K}_1) \times^{\text{in}} \psi_2. \quad (57)$$

Together with the relations asserted by Proposition 11, namely

$$(\phi_{\text{in}}, (\psi_1, \tilde{C}_1) \times^{\text{in}} \psi_2) = (\phi_{\text{in}}, (\psi_1, \tilde{C}_1) \times^{\text{out}} \psi_2) \quad \text{and} \quad (58)$$

$$(\phi_{\text{in}}, (\psi_1, \tilde{K}_1) \times^{\text{in}} \psi_2) = (\phi_{\text{in}}, (\psi_1, \tilde{K}_1) \times^{\text{out}} \psi_2), \quad (59)$$

it follows that

$$(\phi_{\text{in}}, (1 - \varepsilon_M) (\psi_1, \tilde{K}_1) \times^{\text{in}} \psi_2) = 0. \quad (60)$$

This is in conflict with braid group statistics.

In a first step, we show that there exists a path \tilde{C}_1 with the properties (a) through (c). Consider the set J of normalized velocity differences contained in $V_1 \times V_2$,

$$J \doteq \left\{ \frac{v(p_2) - v(p_1)}{\|v(p_2) - v(p_1)\|} : p_i \in V_i \right\},$$

where $v(p) \doteq (1, \mathbf{p}/\omega(\mathbf{p}))$ is the velocity corresponding to p . Similarly, let I be the set of normalized velocity differences contained in $U_1 \times U_2$. I and J are closed intervals in the unit circle of the rest space Σ ,

$$S^1 \doteq \Sigma \cap H.$$

If the sets U_i and V_i are sufficiently small and $V \equiv V_1 + V_2$ is a small neighborhood of some $P \in V_+$, then U_i and V_i are disjoint neighborhoods of points on the circle M_P in P^\perp . Therefore I , $-I$, J and $-J$ are all disjoint. By Lemma A.1, we may assume that C_0 is a *special* space-like cone, $C_0 = \underline{C}_0''$, with small opening angle, centered at some ray $\mathbb{R}^+ e_0$, $e_0 \in S^1$. The path \tilde{C}_0 then corresponds, with the identification (10), to a path \tilde{e}_0 in S^1 ending at e_0 . Now C_0 is future-admissible for $V_1 \times V_2$ (as assumed) if and only if e_0 is in $S^1 \setminus J$, see Eq. (A.1). We wish to construct a special space-like cone C_1 , centered at some unit vector $e \in S^1$, satisfying properties (a) through (c). Again by Eq. (A.1), C_1 satisfies property (a) iff e is in one of the four connected components of $S^1 \setminus \{I \cup -I \cup J \cup -J\}$, and property (b) iff e is contained in the open interval either between I and $-J$ or between $-I$ and J . To be specific, we choose e in the interval between I and $-J$. Define now the path \tilde{e} corresponding to \tilde{C}_1 by appending to \tilde{e}_0 a path from e_0 to e which is contained in $S^1 \setminus J$. Then according to Lemma A.1, \tilde{C}_1 satisfies property (c), too.

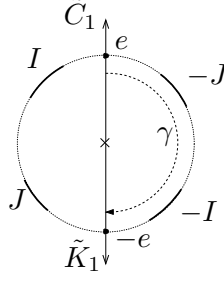


Figure 2: The unit circle in the hyper-plane Σ . C_1 is centered at e and K_1 is centered at $-e$. \tilde{C}_1 and \tilde{K}_1 differ by the path γ .

We now define a second path \tilde{K}_1 as follows. Let K_1 be the special space-like cone $K_1 \doteq -C_1$. It is centered at $-e$, located in the open interval between $-I$ and J , and also satisfies properties (a) and (b). Define the path corresponding to \tilde{K}_1 by appending to \tilde{e} a path γ from e to $-e$ which neither crosses I nor J , see Figure 2. (This is possible since I and J are in the same connected component of $S^1 \setminus \{e, -e\}$.) Then according to Lemma A.1,

$$(\psi_1, \tilde{C}_1) \times^{\text{out}} \psi_2 = (\psi_1, \tilde{K}_1) \times^{\text{out}} \psi_2 \quad (61)$$

holds for all ψ_i with spectral supports in V_i , as well as for all ψ_i with spectral supports in U_i , $i = 1, 2$. (See Figure 4 in the appendix.) In particular, \tilde{K}_1 satisfies property (c), and therefore Prop. 11 applies. But the path γ , which connects \tilde{C}_1 and \tilde{K}_1 , must inevitably cross the interval $-I$. This implies that for ψ_i with spectral supports in U_i the incoming scattering states $(\psi_1, \tilde{K}_1) \times^{\text{in}} \psi_2$ and $(\psi_1, \tilde{C}_1) \times^{\text{in}} \psi_2$ differ by some monodromy operator. To calculate this operator let us assume, to be specific, that the intervals $I, J, -I, -J$ are neighbors in the mathematically positive sense, as in Figure 2. Then our path from C_1 (centered at e) to K_1 (centered at $-e$) must be such that $\theta(\tilde{K}_1) = \theta(\tilde{C}_1) - \pi$ in order not to cross I or J . To calculate the

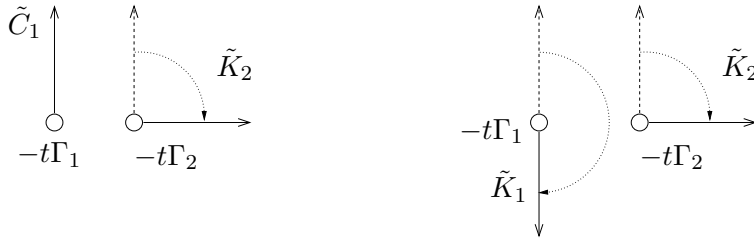


Figure 3: The space-like hyper-plane $\Sigma - tu$, $t > 0$, with $\Gamma_i \doteq \Gamma(U_i)$. (C_1, K_2) and (K_1, K_2) are past-admissible for $U_1 \times U_2$.

behavior of the incoming scattering state under this change, choose a path \tilde{K}_2 such that (C_1, K_2) and (K_1, K_2) are past-admissible for $U_1 \times U_2$ and such that

$$\theta(\tilde{K}_1) < \theta(\tilde{K}_2) < \theta(\tilde{C}_1) \equiv \theta(\tilde{K}_1) + \pi,$$

see Figure 3. Then $\varepsilon_{\pi_1\pi_2}(\tilde{C}_1, \tilde{K}_2) = \varepsilon_{\pi_1\pi_2}$ and $\varepsilon_{\pi_2\pi_1}(\tilde{K}_2, \tilde{K}_1) = \varepsilon_{\pi_2\pi_1}$, hence

$$\begin{aligned} (\psi_1, \tilde{C}_1) \times^{\text{in}} \psi_2 &\equiv (\psi_1, \tilde{C}_1) \times^{\text{in}} (\psi_2, \tilde{K}_2) \\ &= \varepsilon_{\pi_1\pi_2}(\tilde{C}_1, \tilde{K}_2) \varepsilon_{\pi_2\pi_1}(\tilde{K}_2, \tilde{K}_1) (\psi_1, \tilde{K}_1) \times^{\text{in}} (\psi_2, \tilde{K}_2) \\ &= \varepsilon_{\pi_1\pi_2} \varepsilon_{\pi_2\pi_1} (\psi_1, \tilde{K}_1) \times^{\text{in}} \psi_2. \end{aligned}$$

That is, the incoming scattering state in fact picks up the monodromy operator

$$\varepsilon_M \doteq \varepsilon_{\pi_1\pi_2} \varepsilon_{\pi_2\pi_1} \quad (62)$$

under a change from \tilde{C}_1 to \tilde{K}_1 . So Eq. (60) in fact holds, with ε_M as above.

We now show that this is in conflict with braid group statistics. First consider $\pi_1 = \pi_2 = \pi$, where π is an Anyonic sector. Then the monodromy operator equals ω_π^2 times the unit operator, and Eq. (60) implies $\omega_\pi^2 = 1$, since $(\phi_{\text{in}}, (\psi_1, \tilde{K}_1) \times^{\text{in}} \psi_2) \neq 0$ for suitable ϕ_{in} . This proves Theorem 2 in the Abelian case. In the non-Abelian case, where ε_M acts non-trivially, recall that the sector $\pi_1 \times \pi_2$ contains a finite number of irreducible sectors, each with finite multiplicity [18]. For every sub-representation σ contained in $\pi_1 \times \pi_2$, let E_σ be the corresponding projection.¹⁵ By Lemma 3.3 in [18], these projections diagonalize the monodromy operator:

$$E_\sigma \varepsilon_M = \frac{\omega_\sigma}{\omega_1 \omega_2} E_\sigma,$$

where we have written $\omega_i \doteq \omega_{\pi_i}$. Now $\sum_\sigma E_\sigma$, where the finite sum goes over all irreducible sub-representations of $\pi_1 \times \pi_2$, is the unit operator in $\mathcal{H}_{\pi_1 \times \pi_2}$ (and also is a self-intertwiner for $\pi_1 \times \pi_2$). Denoting $\psi_{\text{in}} \doteq (\psi_1, \tilde{K}_1) \times^{\text{in}} \psi_2$, Eq. (60) then implies

$$0 = \sum_\sigma (\phi_{\text{in}}, E_\sigma (1 - \varepsilon_M) \psi_{\text{in}}) = \sum_\sigma \left(1 - \frac{\omega_\sigma}{\omega_1 \omega_2}\right) \chi(E_\sigma) (\phi_{\text{in}}, \psi_{\text{in}}).$$

We have used Lemma 4 and have written $\chi \doteq \chi_1 \chi_2$. Now ψ_{in} is certainly not orthogonal to all allowed ϕ_{in} , and therefore the above equation yields

$$\sum_\sigma \chi(E_\sigma) = \sum_\sigma \frac{\omega_\sigma}{\omega_1 \omega_2} \chi(E_\sigma).$$

But $\chi(E_\sigma)$ is positive, while $\frac{\omega_\sigma}{\omega_1 \omega_2}$ has modulus one. Therefore all factors $\frac{\omega_\sigma}{\omega_1 \omega_2}$ must equal one, and we conclude:

Proposition 12 *For every irreducible sub-representation $\sigma \in \Delta$ of $\pi_1 \times \pi_2$ there holds*

$$\omega_\sigma = \omega_{\pi_1} \omega_{\pi_2}.$$

Let us consider the case $\pi_1 = \pi$ and $\pi_2 = \bar{\pi}$, and recall that the representation $\pi \times \bar{\pi}$ contains the vacuum representation π_0 , which has statistics parameter 1. Since $\omega_{\bar{\pi}}$ and ω_π coincide [19], we conclude that $\omega_\pi^2 = 1$. This concludes the proof of Theorem 2.

¹⁵Choose an orthonormal basis $T_{\sigma,i}$, $i = 1, \dots, N_\sigma$, in the Hilbert space of intertwiners from σ to $\pi_1 \times \pi_2$; Then $E_\sigma \doteq \sum_{i=1}^{N_\sigma} T_{\sigma,i} T_{\sigma,i}^*$.

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A Local independence on the localization regions of scattering states.

For completeness sake, we verify here the well-known fact that the scattering states are *locally* independent of the space-time localization regions \tilde{C} in three-dimensional space-time. Let V_1, V_2 be compact disjoint subsets of the mass shell, and let J be the set of normalized velocity differences contained in $V_1 \times V_2$,

$$J \doteq \left\{ \frac{v(p_2) - v(p_1)}{\|v(p_2) - v(p_1)\|} : p_i \in V_i \right\},$$

where $v(p) \doteq (1, \mathbf{p}/\omega(\mathbf{p}))$ is the velocity corresponding to p . Assuming that V_i are sufficiently small, J is a closed, simply connected, interval in the intersection of the space-like directions H with the rest space Σ ,

$$S^1 \doteq \Sigma \cap H.$$

Let now C_1 be a space-like cone with apex at the origin which is future-admissible for $V_1 \times V_2$. By part *i*) of Lemma 6, this is the case if, and only if, the closure of its set of space-like directions $C_1^H \equiv C_1 \cap H$ is contained in the causal complement of J in H ,¹⁶ which we shall denote by J' and which is also simply connected. Recall from Eq. (10) and thereafter that we have identified a path \tilde{C}_1 of space-like cones ending at C_1 with a sheet in the universal covering space of H over $C_1^H \cong C$, that is, with a set of (homotopy classes of) paths in H ending at C_1^H .

Lemma A.1 *Let \tilde{C}_1 and \tilde{K}_1 be paths ending at respective space-like cones C_1 and K_1 which have their apices at the origin and are future-admissible for $V_1 \times V_2$. If the sheets \tilde{C}_1^H and \tilde{K}_1^H differ by a path in J' , then for all ψ_i with spectral supports in V_i there holds*

$$(\psi_1, \tilde{C}_1) \overset{\text{out}}{\times} \psi_2 = (\psi_1, \tilde{K}_1) \overset{\text{out}}{\times} \psi_2. \quad (\text{A.2})$$

Proof. We first define a localization region C_2 for the state ψ_2 , namely: Let C_2 be a space-like cone with apex at the origin such that C_2^H is contained in the causal completion of J . Then the pairs C_1, C_2 and K_1, C_2 both are future-admissible for $V_1 \times V_2$. Now for any path \tilde{C}_2 ending at C_2 the relative winding number of \tilde{C}_2 with respect to \tilde{C}_1 coincides with that of \tilde{C}_2 w.r.t. \tilde{K}_1 , i.e., $N(\tilde{C}_1, \tilde{C}_2) = N(\tilde{K}_1, \tilde{C}_2)$. Take for example \tilde{C}_2 such that this winding number is -1 , see Figure 4. Then

¹⁶Note that if C_1 is a special space-like cone, $C_1 = (\underline{C}_1)''$ with $\underline{C}_1 \subset \Sigma$, then this condition is equivalent to

$$\overline{\underline{C}_1} \cap J = \emptyset. \quad (\text{A.1})$$

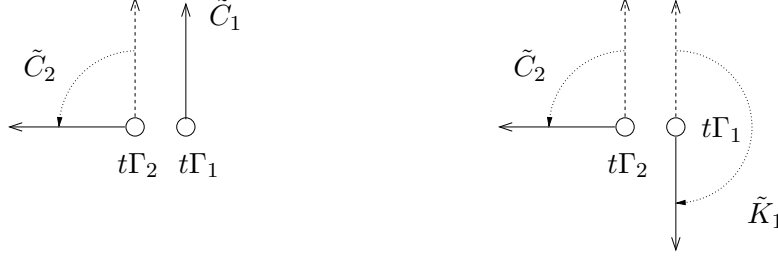


Figure 4: The space-like hyper-plane $\Sigma + tu$, $t > 0$, with $\Gamma_i \doteq \Gamma(U_i)$. (C_1, C_2) and (K_1, C_2) are future-admissible for $U_1 \times U_2$.

$N(\tilde{C}_2, \tilde{K}_1) = 0$ and therefore by Eq. (15) $\varepsilon_{\pi_1\pi_2}(\tilde{C}_1, \tilde{C}_2) = \varepsilon_{\pi_2\pi_1}^{-1}$ and $\varepsilon_{\pi_2\pi_1}(\tilde{C}_2, \tilde{K}_1) = \varepsilon_{\pi_2\pi_1}$. Hence Lemma 5 implies that for $\psi_i \in \mathcal{H}_{\pi_i}$ there holds

$$\begin{aligned} (\psi_1, \tilde{C}_1) \times^{\text{out}} \psi_2 &\equiv (\psi_1, \tilde{C}_1) \times^{\text{out}} (\psi_2, \tilde{C}_2) = \varepsilon_{\pi_1\pi_2}(\tilde{C}_1, \tilde{C}_2) \varepsilon_{\pi_2\pi_1}(\tilde{C}_2, \tilde{K}_1) (\psi_1, \tilde{K}_1) \times^{\text{out}} (\psi_2, \tilde{C}_2) \\ &= (\psi_1, \tilde{K}_1) \times^{\text{out}} (\psi_2, \tilde{C}_2) \equiv (\psi_1, \tilde{K}_1) \times^{\text{out}} \psi_2. \end{aligned}$$

That is, the outgoing scattering state is in fact invariant under a change of localization region from \tilde{C}_1 to \tilde{K}_1 , as claimed. \square

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